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Lax pair for strings in Lunin-Maldacena background

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Abstract: Recently Lunin and Maldacena used an $\mathrm{SL}(3,\mathbb{R})$ transformation of the $\mathrm{AdS}_5 \times S^5$ background to generate a supergravity solution dual to a so-called β-deformation of $\mathcal{N}=4$ super Yang-Mills theory. We use a T-duality-shift-T-duality (TsT) transformation to obtain the β-deformed background for real $\beta \equiv \gamma$, and show that solutions of string theory equations of motion in this background are in one-to-one correspondence with those in $\mathrm{AdS}_5 \times S^5$ with twisted boundary conditions imposed on the U(1) isometry fields. We then apply the TsT transformation to derive a local and periodic Lax pair for the bosonic part of string theory in the γ-deformed background. We also perform a chain of three consecutive TsT transformations to generate a three-parameter deformation of $\mathrm{AdS}_5 \times S^5$. The three-parameter background is dual to a nonsupersymmetric marginal deformation of $\mathcal{N}=4$ SYM. Finally, we combine the TsT transformations with $\mathrm{SL}(2,\mathbb{R})$ ones to obtain a $6+2$ parameter deformation of $\mathrm{AdS}_5 \times S^5$.

Keywords: AdS-CFT and dS-CFT Correspondence, Superstring Vacua.

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JHEP05(2005)069

Contents

1. Introduction

In a recent paper [1], Lunin and Maldacena have found a supergravity background which they have conjectured to be dual to a marginal deformation of $\mathcal{N}=4$ SYM sometimes called a β deformation [2–8].

A relative simplicity of the Lunin-Maldacena supergravity background and the $\mathcal{N}=1$ superconformal theory makes the conjectured duality a new promising arena for studying the AdS/CFT correspondence [9–11]. It is definitely worth to perform tests of the duality by using methods and ideas developed for the simplest example of the AdS/CFT correspondence between superstrings on $AdS_5 \times S^5$ and $\mathcal{N}=4$ SYM. One of such tests has been already performed in [1], where the plane-wave limit of the β-deformed background has been analyzed, and it has been shown that the spectrum of string theory in the pp-wave coincides with the spectrum of BMN-type operators [12] in the β-deformed $\mathcal{N}=4$ SYM [6].

The BMN operators are dual to heavy semiclassical string states which can be analyzed by means of semiclassical methods [13, 14]. It has been found in [15] that there is a large class of multi-spin string solutions in $AdS_5 \times S^5$ whose energies in a special scaling limit admit an expansion in powers of the t' Hooft coupling λ, and, therefore, can be compared with perturbative anomalous dimensions of dual $\mathcal{N}=4$ SYM operators.

Computation of conformal dimensions of "long" $\mathcal{N}=4$ SYM operators became feasible after it was realized in [16] that in the scalar fields sector the one-loop dilatation operator of $\mathcal{N}=4$ SYM coincides with the hamiltonian of an integrable spin chain, and, therefore, one can use the Bethe ansatz for the chain to analyze the spectrum of the dilatation operator. The integrability of the complete one-loop dilatation operator was then demonstrated in [17], and the higher-loop integrability of $\mathcal{N}=4$ SYM was discussed in [18 − 22], see [23] for a review and references.

The Bethe ansatz for the dilatation operator allowed to perform comparisons of field theory and string theory results, first by matching energies of the explicitly known examples of multi-spin string solutions in $AdS_5 \times S^5$ with conformal dimensions of $\mathcal{N}=4$ SYM operators [24 − 29], and then in general by using the hidden relation of the integrable structures of the semiclassical string theory and the spin chain describing long SYM operators [30 − 34], see [35] for a review and references.

It is interesting to understand if similar comparisons can be performed for the Lunin-Maldacena case. It is known that the one-loop dilatation operators in several sectors of the β-deformed $\mathcal{N}=4$ SYM theory can be identified with hamiltonians of integrable spin chains [7, 8]. Thus, anomalous dimensions of primary operators in these sectors can be computed by means of the Bethe ansatz technique.

On the other hand, in the approach of [31], the integrability of string sigma model played a crucial role in the proof of the one- and, especially, two-loop agreement and three-loop disagreement [19] between string theory and gauge theory computations. The Lax representation for superstring theory in $AdS_5 \times S^5$ [36] allows one to derive a system of singular integral equations called string Bethe equations [31, 34] which describes multi-spin string states. At one- and two-loops of SYM perturbation theory the string Bethe equations appear to be equivalent to the thermodynamic limit of the spin chain Bethe equations, thus providing a proof of the matching of multi-spin string energies with conformal dimensions of SYM operators.

Our aim in this paper will be to find a Lax representation for strings in the β-deformed background for real $\beta \equiv \gamma$, which can be used to derive the string Bethe equations.

The β-deformed background was derived from $AdS_5 \times S^5$ by using the $\mathrm{SL}(3,\mathbb{R})$ symmetry of type-IIB supergravity compactified on a 2-torus. On the other hand, if the parameter of the deformation is real, one can obtain the γ-deformed background from $AdS_5 \times S^5$ by means of a T-duality transformation on a U(1) isometry variable φ_1, a shift of another isometry variable, followed by another T-duality on φ_1 [1].[1] We will refer to the chain of these transformations as a TsT-transformation. Then, the solution with complex β is obtained by performing $\mathrm{SL}(2,\mathbb{R})_s$ transformations of the 10-dimensional type IIB supergravity.

In this paper we will discuss the TsT-transformation in detail. In section 2 we rederive the metric and the two-form field part of the Lunin-Maldacena background in the real deformation parameter case by using the TsT-transformation, and show that classical solutions of string theory equations of motion in their background are in one-to-one correspondence with those in the $AdS_5 \times S^5$ background with twisted boundary conditions imposed on the U(1) isometry fields. An interesting property of the twist is that it depends on the conserved U(1) charges of the model, and, therefore, the space of solutions is divided into sectors characterized by the charges. There is no twist if all the three charges

[1] T-duality and shifts were used in [37] to derive Melvin-type solutions from flat space.

JHEP05(2005)069

are equal, $J_1 = J_2 = J_3$, and, therefore, all (J, J, J) string solutions can be obtained from usual periodic string solutions in $AdS_5 \times S^5$.

In section 3 we apply the TsT transformation to derive a local and periodic Lax pair for the bosonic part of string theory in the γ-deformed background.

In section 4 we discuss the Lax representation for the simplest reduction of the string sigma model in the Lunin-Maldacena background to two-spin string states in $R\times S^3_\gamma$. These states are dual to operators from the simplest closed $su(2)_\gamma$ subsector of the γ-deformed $\mathcal{N} = 4$ SYM that consists of operators made out of two holomorphic scalars.

In section 5 we perform a chain of consecutive TsT transformations on each of the three "natural" tori of S^5 with different shift parameters $\hat{\gamma}_i$ to generate a nonsupersymmetric deformation of $AdS_5 \times S^5$. If all the parameters $\hat{\gamma}_i$ are equal to each other the deformation reduces to the Lunin-Maldacena one. The three-parameter supergravity background should be dual to a nonsupersymmetric but marginal deformation of $\mathcal{N} = 4$ SYM, and we discuss the scalar fields part of the deformed YM potential. Since a general Lunin-Maldacena transformation can be symbolically written as $S\,TsT\,S^{-1}$, where S denotes an $\mathrm{SL}(2,\mathbb{R})$ transformation, we also get a 6+2 parameter solution from $AdS_5 \times S^5$ by performing three consecutive STsTS transformations on each of the three tori.

Our conclusions are summarized in section 6.

In appendix A we present the T-duality transformation in the form used in this paper, and in appendix B we write down the $6 + 2$ parameter deformation of $AdS_5 \times S^5$.

2. TsT-transformation

In this section we consider string theory sigma model action on $AdS_5 \times S^5$, and derive the metric and the two-form field part of the Lunin-Maldacena background in the case of real $\beta \equiv \gamma$ by using a T-duality on one circle of S^5, a shift of a second angle variable, followed by another T-duality. It is sufficient to know only the dependence of the string action on g_{mn} and B_{mn} to use it for the semiclassical analysis of multi-spin string states.

Since the TsT-transformation involves only variables of S^5 it is sufficient to consider the S^5 part of the string action that can be written in the form

$$\tilde{\tilde{S}} = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}\left[\gamma^{\alpha\beta}\left(\partial_\alpha r_i \partial_\beta r_i + r_i^2 \partial_\alpha \tilde{\tilde{\phi}}_i \partial_\beta \tilde{\tilde{\phi}}_i\right) + \Lambda(r_i^2 - 1)\right]. \tag{2.1}$$

Here $\sqrt{\lambda} = R^2/\alpha'$, R is the radius of S^5, Λ is a Lagrange multiplier, $i = 1, 2, 3$, and $\gamma^{\alpha\beta} \equiv \sqrt{-h}\, h^{\alpha\beta}$, where $h^{\alpha\beta}$ is a world-sheet metric with Minkowski signature. In the conformal gauge $\gamma^{\alpha\beta} = \mathrm{diag}(-1, 1)$ but we do not fix the world-sheet metric in this section. The action is invariant under the SO(6) group, and the three U(1) isometry transformations are realized as shifts of the angle variables $\tilde{\tilde{\phi}}_i$.

To derive the γ-deformed background it is convenient to make the following change of variables [1]

$$\tilde{\tilde{\phi}}_1 = \tilde{\tilde{\varphi}}_3 - \tilde{\tilde{\varphi}}_2\,, \qquad \tilde{\tilde{\phi}}_2 = \tilde{\tilde{\varphi}}_3 + \tilde{\tilde{\varphi}}_1 + \tilde{\tilde{\varphi}}_2\,, \qquad \tilde{\tilde{\phi}}_3 = \tilde{\tilde{\varphi}}_3 - \tilde{\tilde{\varphi}}_1\,, \qquad \tilde{\tilde{\varphi}}_3 \equiv \tilde{\tilde{\psi}}\,. \tag{2.2}$$

In terms of these new angle variables the action (2.1) takes the form

$$\tilde{\tilde{S}} = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}\left[\gamma^{\alpha\beta}\left(\partial_\alpha r_i \partial_\beta r_i + g_{ij}\,\partial_\alpha\tilde{\tilde{\varphi}}_i\partial_\beta\tilde{\tilde{\varphi}}_j\right) + \Lambda(r_i^2-1)\right], \tag{2.3}$$

where the metric components g_{ij} are

$$\begin{aligned} g_{11} &= r_2^2 + r_3^2\,, & g_{22} &= r_1^2 + r_2^2\,, & g_{33} &= 1\,, \\ g_{12} &= r_2^2\,, & g_{13} &= r_2^2 - r_3^2\,, & g_{23} &= r_2^2 - r_1^2\,. \end{aligned} \tag{2.4}$$

Then we make the T-duality transformation on the circle parameterized by $\tilde{\tilde{\varphi}}_1$. By using the formulas collected in appendix A, we get the action for the T-dual theory

$$\tilde{S} = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}\left[\gamma^{\alpha\beta}\left(\partial_\alpha r_i \partial_\beta r_i + \tilde{g}_{ij}\partial_\alpha\tilde{\varphi}_i\partial_\beta\tilde{\varphi}_j\right) - \epsilon^{\alpha\beta}\tilde{b}_{ij}\partial_\alpha\tilde{\varphi}_i\partial_\beta\tilde{\varphi}_j + \Lambda(r_i^2-1)\right]. \tag{2.5}$$

Here $\epsilon^{01}=1$, the T-t-ransformed metric $\tilde{g}$ and the skew-symmetric B-field $\tilde{b}_{ij}$ are given by

$$\begin{aligned} \tilde{g}_{11} &= \frac{1}{r_2^2+r_3^2}\,, \quad \tilde{g}_{22} = \frac{r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2}{r_2^2+r_3^2}\,, \quad \tilde{g}_{33} = 1 - \frac{(r_2^2-r_3^2)^2}{r_2^2+r_3^2}\,, \quad \tilde{g}_{12} = \tilde{g}_{13} = 0\,, \\ \tilde{g}_{23} &= \frac{2r_2^2r_3^2 - r_1^2r_2^2 - r_1^2r_3^2}{r_2^2+r_3^2}\,, \quad \tilde{b}_{12} = \frac{r_2^2}{r_2^2+r_3^2}\,, \quad \tilde{b}_{13} = \frac{r_2^2-r_3^2}{r_2^2+r_3^2}\,, \quad \tilde{b}_{23} = 0\,. \end{aligned}$$

The T-dual variables $\tilde{\varphi}_i$ are related to $\tilde{\tilde{\varphi}}_i$ as follows

$$\begin{aligned} \epsilon^{\alpha\beta}\partial_\beta\tilde{\varphi}_1 &= \gamma^{\alpha\beta}\partial_\beta\tilde{\tilde{\varphi}}_i\, g_{1i} \quad \Leftrightarrow \quad \partial_\alpha\tilde{\tilde{\varphi}}_1 = \gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\tilde{\varphi}_1\,\tilde{g}_{11} - \partial_\alpha\tilde{\varphi}_i\,\tilde{b}_{1i}\,, \\ \tilde{\tilde{\varphi}}_2 &= \tilde{\varphi}_2\,, \qquad \tilde{\tilde{\varphi}}_3 = \tilde{\varphi}_3\,. \end{aligned} \tag{2.6}$$

The relations (2.6) are satisfied only on-shell, that means that their consistency conditions lead to the equations of motion for $\tilde{\tilde{\varphi}}_i$ and $\tilde{\varphi}_i$.

Next, we make the following shift of the angle variable $\tilde{\varphi}_2$

$$\tilde{\varphi}_2 \to \tilde{\varphi}_2 + \hat{\gamma}\,\tilde{\varphi}_1\,, \tag{2.7}$$

where $\hat{\gamma}$ is any constant. After the shift the T-transformed metric $\tilde{g}$ acquires the following form, $\tilde{g}_{ij} \to \tilde{G}_{ij}$:

$$\begin{aligned} \tilde{G}_{11} &= \tilde{g}_{11} + \hat{\gamma}^2\,\tilde{g}_{22} = \frac{G^{-1}}{r_2^2+r_3^2}\,, \quad G^{-1} = 1 + \hat{\gamma}^2\left(r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2\right), \\ \tilde{G}_{22} &= \tilde{g}_{22}\,, \quad \tilde{G}_{33} = \tilde{g}_{33}\,, \quad \tilde{G}_{12} = \hat{\gamma}\,\tilde{g}_{22}\,, \quad \tilde{G}_{13} = \hat{\gamma}\,\tilde{g}_{23}\,, \quad \tilde{G}_{23} = \tilde{g}_{23}\,, \end{aligned}$$

and the eq. (2.6) transforms into

$$\partial_\alpha\tilde{\tilde{\varphi}}_1 = \gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\tilde{\varphi}_1\,\tilde{g}_{11} - \partial_\alpha\tilde{\varphi}_i\,\tilde{b}_{1i} - \hat{\gamma}\,\partial_\alpha\tilde{\varphi}_1\,\tilde{b}_{12}\,. \tag{2.8}$$

The final step is to make again the T-duality transformation on the circle parameterized by $\tilde{\varphi}_1$. This leads to the string action on the γ-deformed background

$$S = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}\left[\gamma^{\alpha\beta}\left(\partial_\alpha r_i \partial_\beta r_i + G_{ij}\partial_\alpha\varphi_i\partial_\beta\varphi_j\right) - \epsilon^{\alpha\beta}B_{ij}\,\partial_\alpha\varphi_i\partial_\beta\varphi_j + \Lambda(r_i^2-1)\right]. \tag{2.9}$$

The variables $\tilde{\varphi}_i$ are related to the T-dual variables φ_i as follows

$$\begin{aligned} \epsilon^{\alpha\beta}\partial_\beta\varphi_1 &= \gamma^{\alpha\beta}\partial_\beta\tilde{\varphi}_i\,\tilde{G}_{1i} - \epsilon^{\alpha\beta}\partial_\beta\tilde{\varphi}_i\,\tilde{b}_{1i} \quad\Leftrightarrow\quad \partial_\alpha\tilde{\varphi}_1 = \gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\varphi_i\,G_{1i} - \partial_\alpha\varphi_i\,B_{1i}\,, \\ \tilde{\varphi}_2 &= \varphi_2\,, \qquad \tilde{\varphi}_3 = \varphi_3\,. \end{aligned} \tag{2.10}$$

The eqs. (2.6), (2.8) and (2.10) allow us to determine the following relations between the angle variables $\tilde{\tilde{\varphi}}_i$ and the TsT-transformed variables φ_i:

$$\begin{aligned} \partial_\alpha\tilde{\tilde{\varphi}}_1 &= \left(\tilde{g}_{11}G_{1i} + \hat{\gamma}\,\tilde{b}_{12}B_{1i} - \tilde{b}_{1i}\right)\partial_\alpha\varphi_i - \left(\hat{\gamma}\,\tilde{b}_{12}G_{1i} + \tilde{g}_{11}B_{1i}\right)\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\varphi_i\,, \\ \partial_\alpha\tilde{\tilde{\varphi}}_2 &= \partial_\alpha\varphi_2 - \hat{\gamma}\,B_{1i}\partial_\alpha\varphi_i + \hat{\gamma}\,G_{1i}\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\varphi_i\,, \\ \partial_\alpha\tilde{\tilde{\varphi}}_3 &= \partial_\alpha\varphi_3\,. \end{aligned} \tag{2.11}$$

The γ-deformed metric in (2.9) is given by

$$G_{ij} = G\,g_{ij}\,, \qquad \text{if both } i,j \neq 3\,; \qquad G_{33} = G\,g_{33} + 9\,\hat{\gamma}^2\,G\,r_1^2r_2^2r_3^2\,. \tag{2.12}$$

It is easy to see from this form of the metric that in terms of the angle variables ϕ_i, eq. (2.2), the action takes the following simple form

$$\begin{aligned} S = -\frac{\sqrt{\lambda}}{2}\int d\tau\frac{d\sigma}{2\pi}\Bigg[\gamma^{\alpha\beta}\Bigg(&\partial_\alpha r_i\partial_\beta r_i + G\,r_i^2\partial_\alpha\phi_i\partial_\beta\phi_i + \\ &+ \hat{\gamma}^2\,G\,r_1^2r_2^2r_3^2\left(\sum_i\partial_\alpha\phi_i\right)\left(\sum_j\partial_\beta\phi_j\right)\Bigg) - \\ &- 2\,\hat{\gamma}\,G\,\epsilon^{\alpha\beta}\left(r_1^2r_2^2\partial_\alpha\phi_1\partial_\beta\phi_2 + r_2^2r_3^2\partial_\alpha\phi_2\partial_\beta\phi_3 + r_3^2r_1^2\partial_\alpha\phi_3\partial_\beta\phi_1\right) + \\ &+ \Lambda(r_i^2 - 1)\Bigg]\,. \end{aligned} \tag{2.13}$$

It is in this form the gravity background was written in [1].

The relations (2.11) also acquire a very nice and symmetric form being rewritten in terms of ϕ_i:

$$\begin{aligned} \partial_\alpha\tilde{\tilde{\phi}}_1 &= G\left(\partial_\alpha\phi_1 + \hat{\gamma}^2\,r_2^2r_3^2\sum_i\partial_\alpha\phi_i - \hat{\gamma}\,\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\left(r_2^2\partial_\gamma\phi_2 - r_3^2\partial_\gamma\phi_3\right)\right), \\ \partial_\alpha\tilde{\tilde{\phi}}_2 &= G\left(\partial_\alpha\phi_2 + \hat{\gamma}^2\,r_1^2r_3^2\sum_i\partial_\alpha\phi_i - \hat{\gamma}\,\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\left(r_3^2\partial_\gamma\phi_3 - r_1^2\partial_\gamma\phi_1\right)\right), \\ \partial_\alpha\tilde{\tilde{\phi}}_3 &= G\left(\partial_\alpha\phi_3 + \hat{\gamma}^2\,r_1^2r_2^2\sum_i\partial_\alpha\phi_i - \hat{\gamma}\,\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\left(r_1^2\partial_\gamma\phi_1 - r_2^2\partial_\gamma\phi_2\right)\right). \end{aligned} \tag{2.14}$$

In next section we use these relations to find the Lax pair for string theory on the γ-deformed background.

To clarify the meaning of the relations (2.14) we compute the conserved U(1) isometry currents for the string theories on the backgrounds under consideration

$$
\begin{aligned}
\tilde{\tilde{J}}_i^\alpha &= -\sqrt{\lambda}\, r_i^2\, \gamma^{\alpha\beta}\, \partial_\beta \tilde{\tilde{\phi}}_i\,, \\
J_1^\alpha &= -\sqrt{\lambda}\, r_1^2\, \gamma^{\alpha\beta}\, G \left(\partial_\beta \phi_1 + \hat{\gamma}^2\, r_2^2 r_3^2 \sum_i \partial_\beta \phi_i - \hat{\gamma}\, \gamma_{\beta\rho} \epsilon^{\rho\gamma} \left(r_2^2 \partial_\gamma \phi_2 - r_3^2 \partial_\gamma \phi_3 \right) \right), \\
J_2^\alpha &= -\sqrt{\lambda}\, r_2^2\, \gamma^{\alpha\beta}\, G \left(\partial_\beta \phi_2 + \hat{\gamma}^2\, r_1^2 r_3^2 \sum_i \partial_\beta \phi_i - \hat{\gamma}\, \gamma_{\beta\rho} \epsilon^{\rho\gamma} \left(r_3^2 \partial_\gamma \phi_3 - r_1^2 \partial_\gamma \phi_1 \right) \right), \\
J_3^\alpha &= -\sqrt{\lambda}\, r_3^2\, \gamma^{\alpha\beta}\, G \left(\partial_\beta \phi_3 + \hat{\gamma}^2\, r_1^2 r_2^2 \sum_i \partial_\beta \phi_i - \hat{\gamma}\, \gamma_{\beta\rho} \epsilon^{\rho\gamma} \left(r_1^2 \partial_\gamma \phi_1 - r_2^2 \partial_\gamma \phi_2 \right) \right). \qquad (2.15)
\end{aligned}
$$

Comparing the relations (2.14) with the expressions for the currents (2.15), we see that (2.14) is just a statement that the U(1) currents of strings on S^5 are equal to those on the γ-deformed background:

$$\tilde{\tilde{J}}_i^\alpha = J_i^\alpha\,. \qquad (2.16)$$

We expect that the same relations hold for any two backgrounds related by a TsT transformation.

To get more insight into the relations (2.16) let us consider them for the time and space components. The time component J_i^0 of the current J_i^α is just the momentum conjugated to the angle variable ϕ_i, and the charge J_i is equal to the integral of J_i^0 over σ:

$$p_i = J_i^0\,, \quad J_i = \int \frac{d\sigma}{2\pi}\, J_i^0\,, \qquad \tilde{\tilde{p}}_i = \tilde{\tilde{J}}_i^0\,, \quad \tilde{\tilde{J}}_i = \int \frac{d\sigma}{2\pi}\, \tilde{\tilde{J}}_i^0\,. \qquad (2.17)$$

Thus, the time component of the relations (2.16) says that the momenta do not change under the TsT transformation:

$$\tilde{\tilde{p}}_i = p_i\,. \qquad (2.18)$$

To analyze the space component of the relations (2.16), $\tilde{\tilde{J}}_i^1 = J_i^1$, we express the time derivatives $\partial_0 \phi_i \equiv \dot{\phi}_i$ and $\dot{\tilde{\tilde{\phi}}}_i$ through momenta p_i and $\tilde{\tilde{p}}_i$, and substitute them into (2.16). Then the relations $\tilde{\tilde{J}}_i^1 = J_i^1$ take the following simple form

$$
\begin{aligned}
\tilde{\tilde{\phi}}_1' &= \phi_1' + \gamma\, (p_2 - p_3)\,, \\
\tilde{\tilde{\phi}}_2' &= \phi_2' + \gamma\, (p_3 - p_1)\,, \\
\tilde{\tilde{\phi}}_3' &= \phi_3' + \gamma\, (p_1 - p_2)\,, \qquad (2.19)
\end{aligned}
$$

where

$$\gamma = \frac{\hat{\gamma}}{\sqrt{\lambda}}$$

is the deformation parameter that appears in field theory. Taking into account that ϕ_i are angle variables, and integrating (2.19) over σ, we get the following twisted boundary conditions for the U(1) variables $\tilde{\tilde{\phi}}_i$ of the $AdS_5 \times S^5$ theory:

$$\phi_i(2\pi) - \phi_i(0) = 2\pi\, n_i\,, \qquad n_i \text{ are integer winding numbers}\,,$$

$$
\begin{aligned}
\tilde{\tilde{\phi}}_1(2\pi) - \tilde{\tilde{\phi}}_1(0) &= 2\pi\,(n_1 + \gamma\,(J_2 - J_3))\,, \\
\tilde{\tilde{\phi}}_2(2\pi) - \tilde{\tilde{\phi}}_2(0) &= 2\pi\,(n_2 + \gamma\,(J_3 - J_1))\,, \\
\tilde{\tilde{\phi}}_3(2\pi) - \tilde{\tilde{\phi}}_3(0) &= 2\pi\,(n_3 + \gamma\,(J_1 - J_2))\,.
\end{aligned} \tag{2.20}
$$

Since the equations of motion for the U(1) variables have the form $\partial_\alpha J_i^\alpha = 0\,, \partial_\alpha \tilde{\tilde{J}}_i^\alpha = 0$, the relations (2.16) imply that if ϕ_i solve equations of motion for strings in the γ-deformed background then $\tilde{\tilde{\phi}}_i$ solve those in $AdS_5 \times S^5$ with the twisted boundary conditions (2.20) imposed on $\tilde{\tilde{\phi}}_i$, and vice versa. It is not difficult to check that the Virasoro constraints for both models map to each other under the TsT transformation, and, therefore, the energy of such a twisted string in $AdS_5 \times S^5$ is equal to the energy of the corresponding string in the γ-deformed background. The equivalence between closed strings in the γ-deformed background and twisted strings in $AdS_5 \times S^5$ provides an efficient way of finding multi-spin strings on the γ-deformed background by using the known results for the $AdS_5 \times S^5$ case [15, 25]. In particular, many circular string solutions of [15, 25] formally satisfy string equations for noninteger values of winding numbers n_i, and, therefore, the corresponding solutions of string equations for the γ-deformed background can be obtained just by shifting the winding numbers of these circular strings by the twists (2.20). We also see that if $J_1 = J_2 = J_3$ then all the twists vanish, and, therefore, any solution with equal charges $J_1 = J_2 = J_3$ in the γ-deformed model can be obtained from a periodic solution in $AdS_5 \times S^5$.[2] Therefore, according to our discussion above, the energies of that string states in the γ-deformed model and in $AdS_5 \times S^5$ are equal to each other. Note, however, that the solutions are different because given a solution in $AdS_5 \times S^5$ one should still integrate (2.19) to find a solution in the γ-deformed model.

For completeness we rewrite the relations (2.19) in terms of the angle variables φ_i used in [1]. We get

$$
\tilde{\tilde{\varphi}}_1' = \varphi_1' + \gamma\,\pi_2\,, \qquad \tilde{\tilde{\varphi}}_2' = \varphi_2' - \gamma\,\pi_1\,, \qquad \tilde{\tilde{\varphi}}_3' = \varphi_3'\,. \tag{2.21}
$$

Here π_i are momenta conjugated to φ_i. They are related to p_i as follows

$$
\pi_1 = p_2 - p_3\,, \qquad \pi_2 = p_2 - p_1\,, \qquad \pi_3 = p_1 + p_2 + p_3\,.
$$

The twisted boundary conditions for the U(1) variables $\tilde{\tilde{\varphi}}_i$ of the $AdS_5 \times S^5$ theory, therefore, take the form:

$$
\begin{aligned}
\varphi_i(2\pi) - \varphi_i(0) &= 2\pi\, m_i\,, \\
\tilde{\tilde{\varphi}}_1(2\pi) - \tilde{\tilde{\varphi}}_1(0) &= 2\pi\,(m_1 + \gamma\,(J_2 - J_1))\,, \\
\tilde{\tilde{\varphi}}_2(2\pi) - \tilde{\tilde{\varphi}}_2(0) &= 2\pi\,(m_2 + \gamma\,(J_3 - J_2))\,, \\
\tilde{\tilde{\varphi}}_3(2\pi) - \tilde{\tilde{\varphi}}_3(0) &= 2\pi\, m_3\,.
\end{aligned} \tag{2.22}
$$

[2]It is interesting that as was found in [28] the energy of the 3-spin circular string [15] with $J_1 = J_2 = J_3$ matches the conformal dimension of the dual CFT operator up to the 3-loop order. Generally, the agreement persists only up to the 2-loop order [19].

JHEP05(2005)069

3. The Lax pair

In this section we use the known Lax representation for string theory on $AdS_5 \times S^5$, and the relations (2.14) to find a local Lax pair for string theory in the γ-deformed background. Since the AdS_5 part of the string action is not modified by the TsT-transformation we can restrict our attention to the sigma model on S^5.

A convenient parametrization of S^5 is provided by unitary skew-symmetric SU(4) matrices of the form (see, e.g. the second paper of [25]):

$$g = \begin{pmatrix} 0 & X_3 & X_1 & X_2 \\ -X_3 & 0 & X_2^* & -X_1^* \\ -X_1 & -X_2^* & 0 & X_3^* \\ -X_2 & X_1^* & -X_3^* & 0 \end{pmatrix}, \qquad \det g = (|X_1|^2 + |X_2|^2 + |X_3|^2)^2 = 1\,. \tag{3.1}$$

The equations of motion for the sigma model on S^5 follow from the usual action for the principal chiral field:

$$\mathrm{S} = \int \mathrm{d}\tau \mathrm{d}\sigma \gamma^{\alpha\beta} \mathrm{Tr}\Big(g^{-1}\,\partial_\alpha g\, g^{-1}\,\partial_\beta g\Big)\,,$$

and can be written in the form

$$\partial_\alpha(\gamma^{\alpha\beta} R_\beta) = 0\,, \tag{3.2}$$

where we introduce the right current

$$R_\alpha = g^{-1}\,\partial_\alpha g\,.$$

The equations of motion (3.2) are equivalent to the zero curvature condition [38–41]

$$[D_\alpha\,,\, D_\beta] = 0\,, \tag{3.3}$$

where the Lax operator depending on a spectral parameter x is defined as

$$D_\alpha = \partial_\alpha - \frac{R_\alpha^+}{2(x-1)} + \frac{R_\alpha^-}{2(x+1)} \equiv \partial_\alpha - \mathcal{A}_\alpha(x)\,. \tag{3.4}$$

Here the self- and anti-self dual projections of R_α are given by

$$R_\alpha^\pm = (P^\pm)_\alpha^{\ \beta} R_\beta\,, \qquad (P^\pm)_\alpha^{\ \beta} = \delta_\alpha^{\ \beta} \mp \gamma_{\alpha\rho}\epsilon^{\rho\beta}\,. \tag{3.5}$$

The same equations of motion also follow from another Lax operator defined with the help of the left current $L_\alpha = \partial_\alpha g\, g^{-1}$

$$D_\alpha^L = \partial_\alpha - \frac{L_\alpha^+}{2(1-x)} - \frac{L_\alpha^-}{2(1+x)} \equiv \partial_\alpha - \mathcal{A}_\alpha^L(x)\,. \tag{3.6}$$

The two Lax operators are related by the following gauge transformation

$$g^{-1} D_\alpha^L g = \partial_\alpha - \frac{R_\alpha^+}{2(\frac{1}{x}-1)} + \frac{R_\alpha^-}{2(\frac{1}{x}+1)} \equiv \partial_\alpha - \mathcal{A}_\alpha\Big(\frac{1}{x}\Big)\,. \tag{3.7}$$

JHEP05(2005)069

The Lax operator (3.4) for the sigma model on S^5 cannot be used to derive a local Lax operator for the sigma model on the γ-deformed background, because it is not invariant under the U(1) isometry transformations, and, therefore, has an explicit dependence on the angle variables $\tilde{\tilde{\phi}}_i$. The dependence can be easily found if one notices that the matrix g (3.1) can be represented in the following factorized form

$$g(r_i, \tilde{\tilde{\phi}}_i) = M(\tilde{\tilde{\phi}}_i)\, \hat{g}(r_i)\, M(\tilde{\tilde{\phi}}_i)\,, \qquad X_i = r_i\, e^{i\,\tilde{\tilde{\phi}}_i}\,, \tag{3.8}$$

where

$$M(\tilde{\tilde{\phi}}_i) = e^{\tilde{\tilde{\Phi}}}\,, \qquad \tilde{\tilde{\Phi}} = \frac{i}{2}\begin{pmatrix} \tilde{\tilde{\phi}}_1 + \tilde{\tilde{\phi}}_2 + \tilde{\tilde{\phi}}_3 & 0 & 0 & 0 \\ 0 & -\tilde{\tilde{\phi}}_1 - \tilde{\tilde{\phi}}_2 + \tilde{\tilde{\phi}}_3 & 0 & 0 \\ 0 & 0 & \tilde{\tilde{\phi}}_1 - \tilde{\tilde{\phi}}_2 - \tilde{\tilde{\phi}}_3 & 0 \\ 0 & 0 & 0 & -\tilde{\tilde{\phi}}_1 + \tilde{\tilde{\phi}}_2 - \tilde{\tilde{\phi}}_3 \end{pmatrix},$$

$$\hat{g}(r_i) = \begin{pmatrix} 0 & r_3 & r_1 & r_2 \\ -r_3 & 0 & r_2 & -r_1 \\ -r_1 & -r_2 & 0 & r_3 \\ -r_2 & r_1 & -r_3 & 0 \end{pmatrix}, \qquad \hat{g}^{-1} = -\hat{g}\,. \tag{3.9}$$

Using this representation, we get

$$R_\alpha(r_i, \tilde{\tilde{\phi}}_i) = M^{-1}\, \hat{R}_\alpha(r_i, \partial\tilde{\tilde{\phi}}_i)\, M\,, \tag{3.10}$$

where

$$\hat{R}_\alpha(r_i, \partial\tilde{\tilde{\phi}}_i) = \hat{g}^{-1}\partial_\alpha \hat{g} + \hat{g}^{-1}\,\partial_\alpha \tilde{\tilde{\Phi}}\,\hat{g} + \partial_\alpha\tilde{\tilde{\Phi}} = -\hat{g}\partial_\alpha\hat{g} - \hat{g}\,\partial_\alpha\tilde{\tilde{\Phi}}\,\hat{g} + \partial_\alpha\tilde{\tilde{\Phi}}\,. \tag{3.11}$$

It is clear that the dependence of the Lax connection $\mathcal{R}_\alpha$ on the matrix M can be gauged away[3]

$$\begin{aligned} D_\alpha &\to M\, D_\alpha\, M^{-1} = \partial_\alpha - \mathcal{R}_\alpha\,, \\ \mathcal{R}_\alpha &= M\,\mathcal{A}_\alpha\, M^{-1} - M\,\partial_\alpha M^{-1} = \hat{\mathcal{A}}_\alpha + \partial_\alpha\tilde{\tilde{\Phi}}\,. \end{aligned} \tag{3.12}$$

The gauged transformed Lax connection $\mathcal{R}_\alpha$ is, obviously, flat, and is invariant under the U(1) isometries, since it depends on $\tilde{\tilde{\phi}}_i$ only through $\partial_\alpha\tilde{\tilde{\phi}}_i$. Now, to find the Lax representation for the sigma model on the γ-deformed background all one needs to do is to express the angle variables, $\partial_\alpha\tilde{\tilde{\phi}}_i$, of S^5 in terms of the angle variables, $\partial_\alpha\phi_i$, of the γ-deformed background by using the explicit relations (2.14). Since the TsT-transformation maps the equations of motion for the sigma model on S^5 to those on the Lunin-Maldacena background, the zero-curvature condition for the new Lax connection is equivalent to the equations of motion for the sigma model on the γ-deformed background.

[3] The gauge transformation is similar to the one used in [42] to derive a local and periodic Lax connection for the hamiltonian of strings on $AdS_5 \times S^5$.

JHEP05(2005)069

The resulting expression for the Lax connection $\mathcal{R}_\alpha$ is rather complicated, and we refrain from presenting their explicit form. It is interesting, however, to understand the structure of the Lax component $\mathcal{R}_1$, because the monodromy matrix $T(x)$ is defined as its path-ordered exponential. An important property of $\mathcal{R}_1$ is that it does not depend on the world-sheet metric $\gamma^{\alpha\beta}$ if one expresses time derivatives of the fields ϕ_i and r_i through their conjugated momenta. It follows from the fact that

$$\hat{R}_1^{\pm} = \hat{R}_1 \mp \gamma_{1\rho}\epsilon^{\rho\beta}\hat{R}_\beta = \hat{R}_1 \mp \gamma^{0\beta}\hat{R}_\beta = \hat{R}_1 \pm \hat{P}\,, \tag{3.13}$$

where $\hat{P} = -\gamma^{0\beta}\hat{R}_\beta$ is a matrix-valued momentum. The diagonal and off-diagonal parts of $\hat{P}$ determine the momenta of ϕ_i and r_i, respectively. The formula (3.13) allows us to present the Lax component $\mathcal{R}_1$ of (3.12) in the form depending only on the coordinates and their momenta:

$$\mathcal{R}_1 = \partial_1\tilde{\tilde{\Phi}} + \frac{\hat{R}_1}{x^2-1} + \frac{x\,\hat{P}}{x^2-1}\,. \tag{3.14}$$

According to (2.18) and (2.19), the momenta are invariant under the TsT transformation and the coordinates $\tilde{\tilde{\Phi}}$ are just shifted by the momenta. The formula (3.14) can be used to determine the asymptotic behavior of the monodromy matrix around $x = \pm 1$. The result in fact coincides with the one for strings in $R \times S^5$ because all transformations we've done to derive (3.14) do not change the singular terms in the Lax connection at $x = \pm 1$, and, therefore, the asymptotic behavior. A potential problem with this form of $\mathcal{R}_1$ is that it does not vanish at large values of the spectral parameter x, and that may make more difficult to study the large x asymptotic properties of the monodromy matrix. To study the asymptotics it is easier to make an inverse gauge transformation, and use a nonlocal and nonperiodic Lax connection (3.4) with the field g depending on $\tilde{\tilde{\phi}}_i$ which satisfy the twisted boundary conditions (2.19). Note, however, that the monodromy matrix is not similar to the path-ordered exponential of the Lax connection (3.4) because the inverse gauge transformation is not periodic. It would be interesting to analyze the properties of the monodromy matrix and derive the string Bethe equations for the Lunin-Maldacena model analogous to those derived for strings in $AdS_5 \times S^5$ in [31, 34, 42].

4. $su(2)_\gamma$ subsector

The simplest closed subsector of the γ-deformed $\mathcal{N} = 4$ SYM consists of operators made out of two holomorphic scalars. These operators are dual to two-spin string states in $R \times S^3_\gamma$ which is a consistent reduction of the string sigma model on the Lunin-Maldacena background. We will refer to this subsector as the $su(2)_\gamma$ subsector.

The $su(2)_\gamma$ subsector of the γ-deformed model is obtained by setting $r_3 = 0$. It is easy to see that this reduction is compatible with equations of motion. The action (2.13) then simplifies drastically

$$S = -\frac{\sqrt{\lambda}}{2}\int d\tau\frac{d\sigma}{2\pi}\Big[\gamma^{\alpha\beta}(\partial_\alpha r_i\partial_\beta r_i + G\,r_i^2\partial_\alpha\phi_i\partial_\beta\phi_i) - 2\epsilon^{\alpha\beta}\hat{\gamma}\,G\,r_1^2 r_2^2\,\partial_\alpha\phi_1\partial_\beta\phi_2 + \Lambda(r_i^2-1)\Big], \tag{4.1}$$

where $i = 1, 2$.

This action can be obtained from the string action (2.1) on S^3 by means of the same TsT-transformation. The only difference and simplification is that one does not need to make any change of the angle variables $\tilde{\tilde{\phi}}_i$ $(i=1,2)$. One should just make the T-duality on $\tilde{\tilde{\phi}}_1$, the same shift (2.7), and again the T-duality on $\tilde{\phi}_1$. The resulting relations between $\tilde{\tilde{\phi}}_i$ and ϕ_i acquire the following simple form

$$\begin{aligned}
\partial_\alpha \tilde{\tilde{\phi}}_1 &= G\left(\partial_\alpha \phi_1 - \hat{\gamma}\, r_2^2 \gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma \phi_2\right), \\
\partial_\alpha \tilde{\tilde{\phi}}_2 &= G\left(\partial_\alpha \phi_2 + \hat{\gamma}\, r_1^2 \gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma \phi_1\right), \\
G^{-1} &= 1 + \hat{\gamma}^2\, r_1^2 r_2^2\,.
\end{aligned} \tag{4.2}$$

We will use these relations to obtain a local Lax representation for the equations of motion describing the $su(2)_\gamma$ subsector.

In the $su(2)_\gamma$ subsector formulas (2.19) reduce to:

$$\tilde{\tilde{\phi}}_1' = \phi_1' + \gamma\, p_2\,, \qquad \tilde{\tilde{\phi}}_2' = \phi_2' - \gamma\, p_1\,, \tag{4.3}$$

and the twisted boundary conditions (2.20) take the form

$$\begin{aligned}
\phi_i(2\pi) - \phi_i(0) &= 2\pi\, n_i\,, \qquad n_i \text{ are integer winding numbers}, \\
\tilde{\tilde{\phi}}_1(2\pi) - \tilde{\tilde{\phi}}_1(0) &= 2\pi\,(n_1 + \gamma\, J_2)\,, \qquad \tilde{\tilde{\phi}}_2(2\pi) - \tilde{\tilde{\phi}}_2(0) = 2\pi\,(n_2 - \gamma\, J_1)\,.
\end{aligned} \tag{4.4}$$

This shift of the winding numbers n_i is consistent with the spectrum of strings in the Lunin-Maldacena background in the pp-wave limit [1].

In the $su(2)_\gamma$ subsector the Lax connection can be written in terms of 2×2 matrices. We parameterize S^3 by unitary SU(2) matrices of the form:

$$g = \begin{pmatrix} X_1 & X_2 \\ -X_2^* & X_1^* \end{pmatrix}, \qquad \det g = |X_1|^2 + |X_2|^2 = 1\,. \tag{4.5}$$

The Lax operator for the sigma model on S^3 has the same form (3.4). It also has an explicit dependence on the angle variables $\tilde{\tilde{\phi}}_i$. Representing the matrix g (4.5) in the factorized form

$$g(r_i, \tilde{\tilde{\phi}}_i) = \tilde{M}(\tilde{\tilde{\phi}}_i)\, \hat{g}(r_i)\, M(\tilde{\tilde{\phi}}_i)\,, \qquad X_i = r_i\, e^{i\,\tilde{\tilde{\phi}}_i}\,, \tag{4.6}$$

where

$$\begin{aligned}
M &= \begin{pmatrix} 0 & e^{\frac{i}{2}\tilde{\tilde{\phi}}_2} \\ e^{-\frac{i}{2}\tilde{\tilde{\phi}}_2} & 0 \end{pmatrix}\begin{pmatrix} e^{\frac{i}{2}\tilde{\tilde{\phi}}_1} & 0 \\ 0 & e^{-\frac{i}{2}\tilde{\tilde{\phi}}_1} \end{pmatrix} = \begin{pmatrix} 0 & e^{\frac{i}{2}(\tilde{\tilde{\phi}}_2 - \tilde{\tilde{\phi}}_1)} \\ e^{-\frac{i}{2}(\tilde{\tilde{\phi}}_2 - \tilde{\tilde{\phi}}_1)} & 0 \end{pmatrix}, \\
\tilde{M} &= \begin{pmatrix} e^{\frac{i}{2}\tilde{\tilde{\phi}}_1} & 0 \\ 0 & e^{-\frac{i}{2}\tilde{\tilde{\phi}}_1} \end{pmatrix}\begin{pmatrix} 0 & e^{\frac{i}{2}\tilde{\tilde{\phi}}_2} \\ e^{-\frac{i}{2}\tilde{\tilde{\phi}}_2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{\frac{i}{2}(\tilde{\tilde{\phi}}_1 + \tilde{\tilde{\phi}}_2)} \\ e^{-\frac{i}{2}(\tilde{\tilde{\phi}}_1 + \tilde{\tilde{\phi}}_2)} & 0 \end{pmatrix}, \\
\hat{g} &= \begin{pmatrix} r_1 & r_2 \\ -r_2 & r_1 \end{pmatrix}, \quad g^{-1} = g^T\,, \quad M^{-1} = M\,, \quad \tilde{M}^{-1} = \tilde{M}\,,
\end{aligned} \tag{4.7}$$

we get

$$R_\alpha(r_i, \tilde{\tilde{\phi}}_i) = M^{-1}\, \hat{R}_\alpha(r_i, \partial\tilde{\tilde{\phi}}_i)\, M\,,$$

where

$$\hat{R}_\alpha = \frac{i}{2}(\partial_\alpha\tilde{\tilde{\phi}}_2 - \partial_\alpha\tilde{\tilde{\phi}}_1)\,\sigma_3\, +\, \hat{g}^{-1}\partial_\alpha\hat{g}\, - \frac{i}{2}(\partial_\alpha\tilde{\tilde{\phi}}_1 + \partial_\alpha\tilde{\tilde{\phi}}_2)\,\hat{g}^{-1}\,\sigma_3\,\hat{g}\,. \tag{4.8}$$

Gauging away the dependence of the Lax connection $\mathcal{A}_\alpha$ on the matrix, we obtain the Lax connection that depends only on $\partial_\alpha\tilde{\tilde{\phi}}_i$

$$\begin{aligned} D_\alpha \to M\, D_\alpha\, M^{-1} &= \partial_\alpha - \mathcal{R}_\alpha\,, \\ \mathcal{R}_\alpha \,=\, M\,\mathcal{A}_\alpha\, M^{-1} - M\,\partial_\alpha M^{-1} &= \hat{\mathcal{A}}_\alpha + \frac{i}{2}(\partial_\alpha\tilde{\tilde{\phi}}_2 - \partial_\alpha\tilde{\tilde{\phi}}_1)\,\sigma_3 \end{aligned} \tag{4.9}$$

The local Lax representation for the $su(2)_\gamma$ subsector is now obtained by expressing the angle variables, $\partial_\alpha\tilde{\tilde{\phi}}_i$, of S^3 in terms of the angle variables, $\partial_\alpha\phi_i$, of the γ-deformed background by using the explicit relations (4.2).

It is useful to express the Lax component $\mathcal{R}_1$ in terms of the coordinates and their conjugated momenta. Introducing the parametrization

$$r_1 = \cos\theta\,, \qquad r_2 = \sin\theta\,,$$

and the re-scaled momenta (see (2.15))

$$\mathbf{p}_i = -\gamma^{0\alpha}\partial_\alpha\tilde{\tilde{\phi}}_i\,, \qquad \mathbf{p}_\theta = -\gamma^{0\alpha}\partial_\alpha\theta\,,$$

we get the following expressions for the Lax component $\mathcal{R}_1$ of the γ-deformed model

$$\mathcal{R}_1 = i\,\sigma_3\,\mathcal{R}_1^{(3)} + i\,\sigma_1\,\mathcal{R}_1^{(1)} + i\,\sigma_2\,\mathcal{R}_1^{(2)}\,, \tag{4.10}$$

where

$$\begin{aligned} \mathcal{R}_1^{(3)} &= \frac{\tilde{\tilde{\phi}}'_2 - \tilde{\tilde{\phi}}'_1}{2} + \frac{r_2^2\tilde{\tilde{\phi}}'_2 - r_1^2\tilde{\tilde{\phi}}'_1}{x^2-1} + \frac{x(\mathbf{p}_2 - \mathbf{p}_1)}{x^2-1}\,, \\ \mathcal{R}_1^{(1)} &= -\frac{r_1\,r_2(\tilde{\tilde{\phi}}'_1 + \tilde{\tilde{\phi}}'_2)}{x^2-1} - \frac{x(r_2^2\,\mathbf{p}_1 + r_1^2\,\mathbf{p}_2)}{(x^2-1)r_1\,r_2}\,, \\ \mathcal{R}_1^{(2)} &= \frac{x\,\mathbf{p}_\theta + \theta'}{x^2-1}\,, \end{aligned} \tag{4.11}$$

and $\tilde{\tilde{\phi}}'_i$ are expressed through ϕ'_i by using (4.3).

The Lax representation (4.9) for strings in $R\times S^3_\gamma$ has been recently used to derive the string Bethe equations [43]. It has been shown that these equations coincide with the one- and two-loop Bethe equations for the spin chain describing the operators from the holomorphic $su(2)_\gamma$ subsector [7, 8]. This shows that, just as it was for $\mathcal{N}=4$ SYM, in the γ-deformed case there is a perfect match between string theory and field theory results at least for the simplest $su(2)_\gamma$ subsector.

5. Multi-parameter deformations of $AdS_5 \times S^5$

In this section we use a chain of TsT transformations to generate a three-parameter deformation of the $AdS_5 \times S^5$ supergravity background. In the case when all the parameters are equal to each other the deformed background reduces to the one-parameter Lunin-Maldacena background we discussed in the previous sections. At the end of this section we present a 4+2 parameter generalization of the complex β Lunin-Maldacena solution.

We saw in section 2 that to obtain the γ-deformed supergravity background from $AdS_5 \times S^5$ by using a TsT transformation we had to choose a very special torus in S^5. This choice is related to supersymmetry of the Lunin-Maldacena background but in general one may be interested in studying nonsupersymmetric deformations too. In that case, the choice of the torus looks rather superficial. On the other hand, in the parametrization of S^5 we use in (2.1) there are three natural tori: (ϕ_1, ϕ_2), (ϕ_2, ϕ_3) and (ϕ_3, ϕ_1). A TsT transformation applied to any of these tori produces a very simple one-parameter deformation of $AdS_5 \times S^5$ similar to the $su(2)_\gamma$ subsector of the γ-deformed background we discussed in section 4. One may ask how one could get the γ-deformed background by using TsT transformations on the three tori. The answer appears to be very simple. One should just perform a chain of three consecutive TsT transformations on each of the three tori with the same shift parameter $\hat{\gamma}$. If we allow the TsT transformations to have different shift parameters $\hat{\gamma}_i$ we get a nonsupersymmetric deformation of $AdS_5 \times S^5$. The three-parameter supergravity background should be dual to a nonsupersymmetric but marginal deformation of $\mathcal{N} = 4$ SYM.

Since the details of the derivation are very similar to the case of the γ-deformed background we present here only the final results. We apply the first TsT transformation with T-duality acting on the first angle ϕ_1 and the shift parameter equal to $\hat{\gamma}_3$ to the torus (ϕ_1, ϕ_2), then the second TsT transformation with the shift parameter equal to $\hat{\gamma}_1$ to the torus (ϕ_2, ϕ_3), and finally the third TsT transformation with the shift parameter equal to $\hat{\gamma}_2$ to the torus (ϕ_3, ϕ_1).

The resulting type-IIB supergravity background written in string frame takes the form

$$
\begin{aligned}
ds^2_{\rm str} &= R^2 \left[ds^2_{\rm ads} + \sum_i \left(dr_i^2 + G\, r_i^2\, d\phi_i^2 \right) + G\, r_1^2\, r_2^2\, r_3^2 \left(\sum_i \hat{\gamma}_i\, d\phi_i \right)^2 \right] , \qquad (5.1)\\
G^{-1} &= 1 + \hat{\gamma}_3^2\, r_1^2\, r_2^2 + \hat{\gamma}_1^2\, r_2^2\, r_3^2 + \hat{\gamma}_2^2\, r_3^2\, r_1^2 , \qquad R^4 \equiv 4\pi e^{\phi_0} N = \sqrt{\lambda}, \quad \alpha' = 1 ,\\
e^{2\phi} &= e^{2\phi_0}\, G ,\\
B^{NS} &= R^2 G \left(\hat{\gamma}_3\, r_1^2\, r_2^2\, d\phi_1 \wedge d\phi_2 + \hat{\gamma}_1\, r_2^2\, r_3^2\, d\phi_2 \wedge d\phi_3 + \hat{\gamma}_2\, r_3^2\, r_1^2\, d\phi_3 \wedge d\phi_1 \right) ,\\
C_2 &= -4\, R^2\, e^{-\phi_0}\, w_1 \wedge \sum_i \hat{\gamma}_i\, d\phi_i , \qquad \hat{\gamma}_i = R^2 \gamma_i , \quad dw_1 = c_\alpha s_\alpha^3 s_\theta c_\theta\, d\alpha \wedge d\theta ,\\
C_4 &= 4\, R^4\, e^{-\phi_0} \left(w_4 + G\, w_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 \right) ,\\
F_5 &= 4\, R^4\, e^{-\phi_0} \left(\omega_{AdS_5} + G\, \omega_{S^5} \right) , \qquad \omega_{S^5} = dw_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 , \quad \omega_{AdS_5} = dw_4 ,
\end{aligned}
$$

where we used the notations from [1]: $c_\alpha \equiv \cos\alpha$, $s_\alpha \equiv \sin\alpha$, $r_1 = \cos\alpha$, $r_2 = \sin\alpha\cos\theta$, and ω_{S^5} is the volume form of unit radius S^5. We also applied the rules of T-duality[4] for RR fields [44] to find C_2 and C_4.

The relations between the angle variables of the three-parameter deformed background and those of $AdS_5 \times S^5$ are still given by (2.16), and can be used to find the Lax representation for the model in the same way we did in section 3.

Introducing the momenta conjugated to the angle variables, the relations take the form

$$\begin{aligned}
\tilde{\tilde{p}}_i &= p_i \,, \\
\tilde{\tilde{\phi}}'_1 &= \phi'_1 + \gamma_3\, p_2 - \gamma_2\, p_3 \,, \\
\tilde{\tilde{\phi}}'_2 &= \phi'_2 + \gamma_1\, p_3 - \gamma_3\, p_1 \,, \\
\tilde{\tilde{\phi}}'_3 &= \phi'_3 + \gamma_2\, p_1 - \gamma_1\, p_2 \,,
\end{aligned} \tag{5.2}$$

where $\gamma_i = \hat{\gamma}_i/\sqrt{\lambda}$ are the deformation parameters that appear in field theory.

Twisted boundary conditions for the U(1) variables $\tilde{\tilde{\phi}}_i$ of the $AdS_5 \times S^5$ theory take the following form

$$\begin{aligned}
\phi_i(2\pi) - \phi_i(0) &= 2\pi\, n_i \,, \quad n_i \text{ are integer winding numbers} \,, \\
\tilde{\tilde{\phi}}_1(2\pi) - \tilde{\tilde{\phi}}_1(0) &= 2\pi\, (n_1 + \gamma_3\, J_2 - \gamma_2\, J_3) \,, \\
\tilde{\tilde{\phi}}_2(2\pi) - \tilde{\tilde{\phi}}_2(0) &= 2\pi\, (n_2 + \gamma_1\, J_3 - \gamma_3\, J_1) \,, \\
\tilde{\tilde{\phi}}_3(2\pi) - \tilde{\tilde{\phi}}_3(0) &= 2\pi\, (n_3 + \gamma_2\, J_1 - \gamma_1\, J_2) \,.
\end{aligned} \tag{5.3}$$

We see that for generic values of the deformation parameters γ_i all the angular variables have nontrivial twists.

Since the background (5.1) breaks all of the supersymmetry of $AdS_5 \times S^5$, it should be dual to a nonsupersymmetric but marginal deformation of $\mathcal{N} = 4$ SYM.[5] The bosonic part of the deformed YM potential should have the following form[6]

$$\begin{aligned}
V = &\operatorname{Tr}\left[|\Phi_1\Phi_2 - e^{-2i\pi\gamma_3}\Phi_2\Phi_1|^2 + |\Phi_2\Phi_3 - e^{-2i\pi\gamma_1}\Phi_3\Phi_2|^2 + |\Phi_3\Phi_1 - e^{-2i\pi\gamma_2}\Phi_1\Phi_3|^2\right] + \\
&\operatorname{Tr}\left[\left(\left[\Phi_1,\, \bar{\Phi}_1\right] + \left[\Phi_2,\, \bar{\Phi}_2\right] + \left[\Phi_3,\, \bar{\Phi}_3\right]\right)^2\right] ,
\end{aligned} \tag{5.4}$$

where Φ_i are the three holomorphic scalars of $\mathcal{N} = 4$ SYM. The potential can be obtained from the undeformed one by replacing the usual product $\Phi_i\Phi_j$ by the associative $*$-product of [1]. It should be possible to obtain the fermionic part of the potential by the same procedure, and to check if the deformation is marginal at one-loop. It is known that the one-loop dilatation operator associated to the potential (5.4) is described by integrable spin chains in the $su(2)_{\gamma_i}$[7] and $su(3)_{\gamma_i}$ subsectors of the deformed YM model [7]. The existence of the Lax pair representation for the bosonic part of the string sigma model on the three-parameter deformed background implies that the sigma model is integrable too.

[4]Note that the standard signs of the B^{NS} and C_2 T-duality rules were flipped in [1]. The T-duality rules (A.7) match the choice made in [1]. The author thanks Oleg Lunin for a discussion of this point.

[5]It is unclear if the deformation is marginal for finite N.

[6]The deformation is not supersymmetric and cannot be written in terms of $\mathcal{N} = 1$ superfields.

[7]The $su(2)_{\gamma_i}$ subsector coincides with the $su(2)_\gamma$ subsector of the Lunin-Maldacena model.

It would be interesting to find the Bethe ansatz [45] for the $su(3)_{\gamma_i}$ spin chain and string Bethe equations for the deformed background.

The three-parameter deformed background (5.1) has the same structure as the γ-deformed one. In principle one can use $\mathrm{SL}(2,\mathbb{R})_s$ transformations to generate more general solutions [1] similar to the Lunin-Maldacena background with complex β. In fact, since a general Lunin-Maldacena transformation can be symbolically written as $S_\sigma T s_\gamma T S_\sigma^{-1}$, where S_σ denotes an $\mathrm{SL}(2,\mathbb{R})_s$ transformation with a parameter σ_s, and $T s_\gamma T$ denotes a TsT transformation with the shift parameter γ, we can get a 6+2 parameter solution from $AdS_5 \times S^5$ by performing three consecutive STsTS transformations on each of the three tori. Let us also mention that the step involving S-duality departs from the world sheet treatment, see [43] for a detailed discussion. The 6+2 parameter solution appears to be rather complicated, in particular, it has nonvanishing $G_{\phi_i\alpha}$ and $G_{\phi_i\theta}$ components of the metric. Its explicit form is given in appendix B. Here we present a simpler solution with 4+2 parameters corresponding to the three γ_i, one σ_s, the dilaton ϕ_0 and the axion χ_0. The solution generalizes the general supersymmetry preserving Lunin-Maldacena deformation of $AdS_5 \times S^5$. It is worth noting that the 4+2 parameter solution cannot be obtained by using just one $\mathrm{SL}(3,\mathbb{R})$ transformation of $AdS_5 \times S^5$. It is generated by performing the transformation $S_\sigma T s T_{\gamma_1} T s T_{\gamma_2} T s T_{\gamma_3} S_\sigma^{-1}$.

To derive the solution we begin with the $AdS_5 \times S^5$ background with constant dilaton ϕ_0 and axion χ, and perform the following $\mathrm{SL}(2,\mathbb{R})$ transformation of the background

$$
\begin{aligned}
&\chi + i\, e^{-\phi_0} \equiv \tau \to \tau_{\sigma_s} = \frac{\tau}{1-\sigma_s\,\tau}\,, \qquad B^{NS} \to B^{NS}_{\sigma_s} = B^{NS} - \sigma_s\, C_2\,,\\
&C_4 \to C_4^{\sigma_s} = C_4 - \frac{1}{2}\,\sigma_s\, C_2 \wedge C_2\,.
\end{aligned}
\tag{5.5}
$$

The Einstein frame metric and the two-form C_2 remain invariant under this transformation. It is worth mentioning that the parameter σ_s is not equal to the parameter σ used in [1].

Then we perform the same chain of TsT transformations we used to generate the three-parameter deformed background (5.1). Finally, we perform the inverse $\mathrm{SL}(2,\mathbb{R})$ transformation with σ_s replace by $-\sigma_s$ in (5.5).

The resulting 4+2 parameter type-IIB supergravity background written in string frame takes the form

$$
\begin{aligned}
ds^2_{\rm str} &= R^2\, H^{1/2}\left[ds^2_{\rm ads} + \sum_i \left(dr_i^2 + G\, r_i^2\, d\phi_i^2\right) + G\, r_1^2\, r_2^2\, r_3^2 \left|\sum_i \tilde\beta_i\, d\phi_i\right|^2\right],\\
e^{2\phi} &= e^{2\phi_0}\, G\, H^2\,, \qquad \chi = \chi_0 + e^{-\phi_0}\, H^{-1} Q\,,\\
B^{NS} &= R^2 \left(G\, w_B - 4\, w_1 \wedge \sum_i \tilde\sigma_i\, d\phi_i\right),\\
C_2 &= R^2 \left(G\, w_C - 4\, w_1 \wedge \sum_i (e^{-\phi_0}\tilde\gamma_i + \chi_0 \tilde\sigma_i)\, d\phi_i\right),\\
C_4 &= 4\, R^4\, e^{-\phi_0} \left(w_4 + G(H - e^{\phi_0}\chi_0 Q)\, w_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3\right),\\
F_5 &= 4\, R^4\, e^{-\phi_0}\, (\omega_{AdS_5} + G\, \omega_{S^5})\,.
\end{aligned}
\tag{5.6}
$$

Here the functions G, H, and Q, and the parameters $\tilde\beta_i$, $\tilde\gamma_i$ and $\tilde\sigma_i$ are defined as follows

$$
\begin{aligned}
G^{-1} &= 1 + |\tilde\beta_3|^2\, r_1^2\, r_2^2 + |\tilde\beta_1|^2\, r_2^2\, r_3^2 + |\tilde\beta_2|^2\, r_3^2\, r_1^2\,,\\
H &= 1 + \tilde\sigma_3^2\, r_1^2\, r_2^2 + \tilde\sigma_1^2\, r_2^2\, r_3^2 + \tilde\sigma_2^2\, r_3^2\, r_1^2\,,\\
Q &= \tilde\sigma_3\tilde\gamma_3\, r_1^2\, r_2^2 + \tilde\sigma_1\tilde\gamma_1\, r_2^2\, r_3^2 + \tilde\sigma_2\tilde\gamma_2\, r_3^2\, r_1^2\,,\\
w_B &= \tilde\gamma_3\, r_1^2\, r_2^2\, d\phi_1\wedge d\phi_2 + \tilde\gamma_1\, r_2^2\, r_3^2\, d\phi_2\wedge d\phi_3 + \tilde\gamma_2\, r_3^2\, r_1^2\, d\phi_3\wedge d\phi_1\,,\\
w_C &= \chi_0\, w_B - e^{-\phi_0}\left(\tilde\sigma_3\, r_1^2\, r_2^2\, d\phi_1\wedge d\phi_2 + \tilde\sigma_1\, r_2^2\, r_3^2\, d\phi_2\wedge d\phi_3 + \tilde\sigma_2\, r_3^2\, r_1^2\, d\phi_3\wedge d\phi_1\right),\\
\tilde\gamma_i &= R^2\gamma_i(1-\chi_0\sigma_s) = \hat\gamma_i(1-\chi_0\sigma_s)\,,\quad \tilde\sigma_i = R^2 e^{-\phi_0}\gamma_i\sigma_s = \hat\gamma_i e^{-\phi_0}\sigma_s\,,\quad \tilde\beta_i = \tilde\gamma_i - i\tilde\sigma_i\,.
\end{aligned}
\tag{5.7}
$$

It is easy to see that if all γ_i are equal to each other then the background coincides with the Lunin-Maldacena one provided the parameter σ in [1] is related to σ_s as follows

$$\sigma = \sigma_s\gamma\,, \qquad \gamma_i = \gamma\,.$$

The 4+2 parameter background should be dual to a nonsupersymmetric marginal (at the large-N limit) deformation of $\mathcal{N}=4$ SYM. It would be interesting to find this dual nonsupersymmetric YM model.

6. Conclusion

In this paper we have discussed the TsT transformation of the $AdS_5\times S^5$ background, and shown how it can be used to generate the Lunin-Maldacena supergravity solution in the case of the real deformation parameter γ. We have used the TsT transformation to find the relation between the angle variables of $AdS_5\times S^5$, and the angle variables of the γ-deformed background, and used the relation to derive a local and periodic Lax representation for the γ-deformed model. The existence of the Lax pair implies the integrability of the (bosonic part of) string sigma model on the γ-deformed background.

It is clear that it should be possible to apply the TsT transformation to the Green-Schwarz κ-symmetric superstring action on $AdS_5\times S^5$ [46] to generate the γ-deformed background with all the supergravity fields included. To this end one can try to use the rules of T-duality formulated in [47] for the Green-Schwarz superstring. Then, the approach used in section 3 of this paper should lead to a local and periodic Lax representation for the complete Green-Schwarz sigma model on the γ-deformed supergravity background.

It would be interesting to use the Lax representation to analyze the properties of the monodromy matrix and derive the string Bethe equations for the Lunin-Maldacena model analogous to those derived for strings on $AdS_5\times S^5$ in [31, 34, 42]. The string Bethe equations could be then compared with the thermodynamic limit of the Bethe equations for the γ-deformed $\mathcal{N}=4$ SYM theory [7, 8]. It has been already done for the simplest $su(2)_\gamma$ case in [43].

As another interesting application of the TsT transformation we generated the three-parameter regular supergravity background by using a chain of TsT transformations applied to different tori of $AdS_5\times S^5$. This background is expected to be dual to a nonsupersymmetric marginal deformation of $\mathcal{N}=4$ SYM theory. It should be possible to perform a detailed analysis of the three-parameter background, and the dual conformal field theory.

JHEP05(2005)069

We also derived a $6+2$ parameter deformation of $AdS_5 \times S^5$ by applying a chain of STsTS transformations to the three tori of S^5. The type-IIB supergravity solution is nonsupersymmetric, and it is important to check if it is perturbatively stable. Since it depends on the continuous deformation parameters one might expect that the background is stable at least for small values of the parameters. It also would be interesting to find the nonsupersymmetric conformal deformation of $\mathcal{N}=4$ SYM model dual to this $6+2$ parameter background.

It is of interest to generate other multi-parameter regular backgrounds by using a chain of the TsT and STsTS transformations applied to $AdS_5 \times S^5$, and other supergravity backgrounds with $\mathrm{U}(1)^3$ isometry. In particular, one can consider nonsupersymmetric marginal deformations of theories based on toric manifolds [49–51]. One can also use the TsT transformations to derive nonsupersymmetric deformations of supergravity backgrounds dual to nonconformal field theories such as the Klebanov-Strassler background [52].

In general, the Lunin-Maldacena type backgrounds and the TsT (STsTS) transformation have many interesting properties which are worth studying.

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A. T-duality transformation

In this appendix we present the T-duality transformation [48] in the form used in the paper. We start with the following string theory action

$$S=-\frac{\sqrt{\lambda}}{2}\int d\tau\frac{d\sigma}{2\pi}\left[\gamma^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N\,G_{MN}(X^i)-\epsilon^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N\,B_{MN}(X^i)\right]. \tag{A.1}$$

Here $\epsilon^{01}=1$, $M=1,2,3\dots$, $i=2,3\dots$, and the background fields G_{MN} and B_{MN} do not depend on X^1. The action can be represented in the following equivalent form

$$\begin{aligned}S=-\sqrt{\lambda}\int d\tau\frac{d\sigma}{2\pi}\Bigg[&p^\alpha\left(\partial_\alpha X^M\frac{G_{1M}}{G_{11}}-\gamma_{\alpha\beta}\epsilon^{\beta\rho}\partial_\rho X^M\frac{B_{1M}}{G_{11}}\right)-\frac{1}{2\,G_{11}}\gamma_{\alpha\beta}\,p^\alpha p^\beta+\\&+\frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j\left(G_{ij}-\frac{G_{1i}G_{1j}-B_{1i}B_{1j}}{G_{11}}\right)-\\&-\frac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j\left(B_{ij}-\frac{G_{1i}B_{1j}-B_{1i}G_{1j}}{G_{11}}\right)\Bigg].\end{aligned} \tag{A.2}$$

Indeed, variating with respect to p^α, one gets the following equation of motion for p^α

$$p^\alpha=\gamma^{\alpha\beta}\partial_\beta X^M G_{1M}-\epsilon^{\alpha\beta}\partial_\beta X^M B_{1M}\,. \tag{A.3}$$

Substituting (A.3) into (A.2) and using the identity $\epsilon^{\alpha\gamma}\gamma_{\gamma\rho}\epsilon^{\rho\beta}=\gamma^{\alpha\beta}$, we reproduce the action (A.1).

On the other hand, variating (A.2) with respect to X^1 gives

$$\partial_\alpha\, p^\alpha = 0\,. \tag{A.4}$$

The general solution to this equation can be written in the form

$$p^\alpha = \epsilon^{\alpha\beta}\partial_\beta \tilde{X}^1\,, \tag{A.5}$$

where $\tilde{X}^1$ is the scalar T-dual to X^1. Substituting (A.5) into the action (A.2), we obtain the following T-dual action

$$\tilde{S} = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}\left[\gamma^{\alpha\beta}\partial_\alpha \tilde{X}^M \partial_\beta \tilde{X}^N\, \tilde{G}_{MN} - \epsilon^{\alpha\beta}\partial_\alpha \tilde{X}^M \partial_\beta \tilde{X}^N\, \tilde{B}_{MN}\right], \tag{A.6}$$

where

$$\begin{aligned}
\tilde{G}_{11} &= \frac{1}{G_{11}}\,, \qquad \tilde{G}_{ij} = G_{ij} - \frac{G_{1i}G_{1j} - B_{1i}B_{1j}}{G_{11}}\,, \qquad \tilde{G}_{1i} = \frac{B_{1i}}{G_{11}}\,,\\
\tilde{B}_{ij} &= B_{ij} - \frac{G_{1i}B_{1j} - B_{1i}G_{1j}}{G_{11}}\,, \qquad \tilde{B}_{1i} = \frac{G_{1i}}{G_{11}}\,,\\
\epsilon^{\alpha\beta}\partial_\beta \tilde{X}^1 &= \gamma^{\alpha\beta}\partial_\beta X^M G_{1M} - \epsilon^{\alpha\beta}\partial_\beta X^M B_{1M}\,, \qquad \tilde{X}^i = X^i\,.
\end{aligned} \tag{A.7}$$

B. $6+2$ parameter background

The 6+2 parameter solution can be obtained by performing the transformation

$$S_{\sigma_1^s} T s T_{\gamma_1} S_{\sigma_2^s - \sigma_1^s} T s T_{\gamma_2} S_{\sigma_3^s - \sigma_2^s} T s T_{\gamma_3} S_{-\sigma_3^s}\,,$$

that is a chain of the three consecutive STsTS transformations on each of the three tori.

The resulting 6+2 parameter type-IIB supergravity background written in string frame takes the form

$$\begin{aligned}
ds^2_{\rm str} &= R^2 H^{1/2}\Big[ds^2_{\rm ads} + \sum_i \left(dr_i^2 + G\, r_i^2\, d\phi_i^2\right) + G\, r_1^2\, r_2^2\, r_3^2 \Big|\sum_i \tilde{\beta}_i\, d\phi_i\Big|^2\\
&\qquad + 8\, G\, w_1 \Big((\hat{\gamma}_3\tilde{\sigma}_2 - \hat{\gamma}_2\tilde{\sigma}_3) r_1^2 d\phi_1 + (\hat{\gamma}_1\tilde{\sigma}_3 - \hat{\gamma}_3\tilde{\sigma}_1) r_2^2 d\phi_2 +\\
&\qquad\qquad + (\hat{\gamma}_2\tilde{\sigma}_1 - \hat{\gamma}_1\tilde{\sigma}_2) r_3^2 d\phi_3\Big)\\
&\qquad + 16\, G\, w_1^2 \Big((\hat{\gamma}_3\tilde{\sigma}_2 - \hat{\gamma}_2\tilde{\sigma}_3)^2 r_1^2 + (\hat{\gamma}_1\tilde{\sigma}_3 - \hat{\gamma}_3\tilde{\sigma}_1)^2 r_2^2 + (\hat{\gamma}_2\tilde{\sigma}_1 - \hat{\gamma}_1\tilde{\sigma}_2)^2 r_3^2\Big)\Big]\,,\\
e^{2\phi} &= e^{2\phi_0}\, G\, H^2\,, \qquad \chi = \chi_0 + e^{-\phi_0}\, H^{-1} Q\,,\\
B^{NS} &= R^2\, G\, (w_B - 4\, w_1 \wedge A_B)\,,\\
C_2 &= R^2\, G\left(w_C - 4\, e^{-\phi_0}\, w_1 \wedge A_C\right),\\
C_4 &= 4\, R^4\, e^{-\phi_0}\left(w_4 + G(H - e^{\phi_0}\chi_0 Q)\, w_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3\right),\\
F_5 &= 4\, R^4\, e^{-\phi_0}\, (\omega_{AdS_5} + G\, \omega_{S^5})\,.
\end{aligned} \tag{B.1}$$

JHEP05(2005)069

Here the functions G, H, and Q, and the parameters $\tilde{\beta}_i$, $\tilde{\gamma}_i$ and $\tilde{\sigma}_i$ are defined as follows

$$
\begin{aligned}
G^{-1} &= 1 + |\tilde{\beta}_3|^2\, r_1^2\, r_2^2 + |\tilde{\beta}_1|^2\, r_2^2\, r_3^2 + |\tilde{\beta}_2|^2\, r_3^2\, r_1^2 \\
&\quad + r_1^2 r_2^2 r_3^2 \left((\hat{\gamma}_3\tilde{\sigma}_2 - \hat{\gamma}_2\tilde{\sigma}_3)^2 r_1^2 + (\hat{\gamma}_1\tilde{\sigma}_3 - \hat{\gamma}_3\tilde{\sigma}_1)^2 r_2^2 + (\hat{\gamma}_2\tilde{\sigma}_1 - \hat{\gamma}_1\tilde{\sigma}_2)^2 r_3^2 \right), \\
H &= 1 + \tilde{\sigma}_3^2\, r_1^2\, r_2^2 + \tilde{\sigma}_1^2\, r_2^2\, r_3^2 + \tilde{\sigma}_2^2\, r_3^2\, r_1^2\,, \\
Q &= \tilde{\sigma}_3\tilde{\gamma}_3\, r_1^2\, r_2^2 + \tilde{\sigma}_1\tilde{\gamma}_1\, r_2^2\, r_3^2 + \tilde{\sigma}_2\tilde{\gamma}_2\, r_3^2\, r_1^2\,, \\
w_B &= \left[\tilde{\gamma}_3\, r_1^2\, r_2^2 + r_1^2 r_2^2 r_3^2 \left(\tilde{\sigma}_1(\hat{\gamma}_3\tilde{\sigma}_1 - \hat{\gamma}_1\tilde{\sigma}_3) r_2^2 + \tilde{\sigma}_2(\hat{\gamma}_3\tilde{\sigma}_2 - \hat{\gamma}_2\tilde{\sigma}_3) r_1^2\right)\right] d\phi_1 \wedge d\phi_2 + \\
&\quad + \left[\tilde{\gamma}_1\, r_2^2\, r_3^2 + r_1^2 r_2^2 r_3^2 \left(\tilde{\sigma}_2(\hat{\gamma}_1\tilde{\sigma}_2 - \hat{\gamma}_2\tilde{\sigma}_1) r_3^2 + \tilde{\sigma}_3(\hat{\gamma}_1\tilde{\sigma}_3 - \hat{\gamma}_3\tilde{\sigma}_1) r_2^2\right)\right] d\phi_2 \wedge d\phi_3 + \\
&\quad + \left[\tilde{\gamma}_2\, r_3^2\, r_1^2 + r_1^2 r_2^2 r_3^2 \left(\tilde{\sigma}_3(\hat{\gamma}_2\tilde{\sigma}_3 - \hat{\gamma}_3\tilde{\sigma}_2) r_1^2 + \tilde{\sigma}_1(\hat{\gamma}_2\tilde{\sigma}_1 - \hat{\gamma}_1\tilde{\sigma}_2) r_3^2\right)\right] d\phi_3 \wedge d\phi_1\,, \\
A_B &= \left[\tilde{\sigma}_1 + \tilde{\sigma}_1(\tilde{\gamma}_1^2 + \tilde{\sigma}_1^2) r_2^2 r_3^2 + \tilde{\sigma}_2(\tilde{\gamma}_1\tilde{\gamma}_2 + \tilde{\sigma}_1\tilde{\sigma}_2) r_3^2 r_1^2 + \tilde{\sigma}_3(\tilde{\gamma}_1\tilde{\gamma}_3 + \tilde{\sigma}_1\tilde{\sigma}_3) r_1^2 r_2^2\right] d\phi_1 + \\
&\quad + \left[\tilde{\sigma}_2 + \tilde{\sigma}_2(\tilde{\gamma}_2^2 + \tilde{\sigma}_2^2) r_3^2 r_1^2 + \tilde{\sigma}_1(\tilde{\gamma}_1\tilde{\gamma}_2 + \tilde{\sigma}_1\tilde{\sigma}_2) r_2^2 r_3^2 + \tilde{\sigma}_3(\tilde{\gamma}_2\tilde{\gamma}_3 + \tilde{\sigma}_2\tilde{\sigma}_3) r_1^2 r_2^2\right] d\phi_2 + \\
&\quad + \left[\tilde{\sigma}_3 + \tilde{\sigma}_3(\tilde{\gamma}_3^2 + \tilde{\sigma}_3^2) r_1^2 r_2^2 + \tilde{\sigma}_1(\tilde{\gamma}_1\tilde{\gamma}_3 + \tilde{\sigma}_1\tilde{\sigma}_3) r_2^2 r_3^2 + \tilde{\sigma}_2(\tilde{\gamma}_2\tilde{\gamma}_3 + \tilde{\sigma}_2\tilde{\sigma}_3) r_3^2 r_1^2\right] d\phi_3\,, \\
w_C &= \Big[(\chi\tilde{\gamma}_3 - e^{-\phi_0}\tilde{\sigma}_3) r_1^2 r_2^2 + \\
&\qquad + e^{-\phi_0} r_1^2 r_2^2 r_3^2 \left(\hat{\gamma}_1(\hat{\gamma}_3\tilde{\sigma}_1 - \hat{\gamma}_1\tilde{\sigma}_3) r_2^2 + \hat{\gamma}_2(\hat{\gamma}_3\tilde{\sigma}_2 - \hat{\gamma}_2\tilde{\sigma}_3) r_1^2\right)\Big] d\phi_1 \wedge d\phi_2 + \\
&\quad + \Big[(\chi\tilde{\gamma}_1 - e^{-\phi_0}\tilde{\sigma}_1) r_2^2 r_3^2 + \\
&\qquad + e^{-\phi_0} r_1^2 r_2^2 r_3^2 \left(\hat{\gamma}_2(\hat{\gamma}_1\tilde{\sigma}_2 - \hat{\gamma}_2\tilde{\sigma}_1) r_3^2 + \hat{\gamma}_3(\hat{\gamma}_1\tilde{\sigma}_3 - \hat{\gamma}_3\tilde{\sigma}_1) r_2^2\right)\Big] d\phi_2 \wedge d\phi_3 + \\
&\quad + \Big[(\chi\tilde{\gamma}_2 - e^{-\phi_0}\tilde{\sigma}_2) r_3^2 r_1^2 + \\
&\qquad + e^{-\phi_0} r_1^2 r_2^2 r_3^2 \left(\hat{\gamma}_3(\hat{\gamma}_2\tilde{\sigma}_3 - \hat{\gamma}_3\tilde{\sigma}_2) r_1^2 + \hat{\gamma}_1(\hat{\gamma}_2\tilde{\sigma}_1 - \hat{\gamma}_1\tilde{\sigma}_2) r_3^2\right)\Big] d\phi_3 \wedge d\phi_1\,, \qquad \text{(B.2)} \\
A_C &= \left[\hat{\gamma}_1 + \hat{\gamma}_1(\tilde{\gamma}_1^2 + \tilde{\sigma}_1^2) r_2^2 r_3^2 + \hat{\gamma}_2(\tilde{\gamma}_1\tilde{\gamma}_2 + \tilde{\sigma}_1\tilde{\sigma}_2) r_3^2 r_1^2 + \hat{\gamma}_3(\tilde{\gamma}_1\tilde{\gamma}_3 + \tilde{\sigma}_1\tilde{\sigma}_3) r_1^2 r_2^2\right] d\phi_1 + \\
&\quad + \left[\hat{\gamma}_2 + \hat{\gamma}_2(\tilde{\gamma}_2^2 + \tilde{\sigma}_2^2) r_3^2 r_1^2 + \hat{\gamma}_1(\tilde{\gamma}_1\tilde{\gamma}_2 + \tilde{\sigma}_1\tilde{\sigma}_2) r_2^2 r_3^2 + \hat{\gamma}_3(\tilde{\gamma}_2\tilde{\gamma}_3 + \tilde{\sigma}_2\tilde{\sigma}_3) r_1^2 r_2^2\right] d\phi_2 + \\
&\quad + \left[\hat{\gamma}_3 + \hat{\gamma}_3(\tilde{\gamma}_3^2 + \tilde{\sigma}_3^2) r_1^2 r_2^2 + \hat{\gamma}_1(\tilde{\gamma}_1\tilde{\gamma}_3 + \tilde{\sigma}_1\tilde{\sigma}_3) r_2^2 r_3^2 + \hat{\gamma}_2(\tilde{\gamma}_2\tilde{\gamma}_3 + \tilde{\sigma}_2\tilde{\sigma}_3) r_3^2 r_1^2\right] d\phi_3\,, \\
\tilde{\gamma}_i &= R^2\gamma_i(1 - \chi_0\sigma_i^s) = \hat{\gamma}_i(1 - \chi_0\sigma_i^s)\,, \quad \tilde{\sigma}_i = R^2 e^{-\phi_0}\gamma_i\sigma_i^s = \hat{\gamma}_i e^{-\phi_0}\sigma_i^s\,, \quad \tilde{\beta}_i = \tilde{\gamma}_i - i\tilde{\sigma}_i\,.
\end{aligned}
$$

The self-dual five-form F_5 is expressed in terms of C_4, C_2 and B as follows

$$F_5 = dC_4 - dB^{NS} \wedge C_2\,.$$

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